# THE CLASSIFICATION OF FOOTBALL PATTERNS

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ABSTRACT. We prove that every spherical football is a branched cover, branched only in the vertices, of the standard football made up of 12 pentagons and 20 hexagons. We also give examples showing that the corresponding result is not true for footballs of higher genera. Moreover, we classify the possible pairs (k, l) for which football patterns on the sphere exist satisfying a natural generalisation of the usual incidence relation between pentagons and hexagons to k-gons and l-gons.
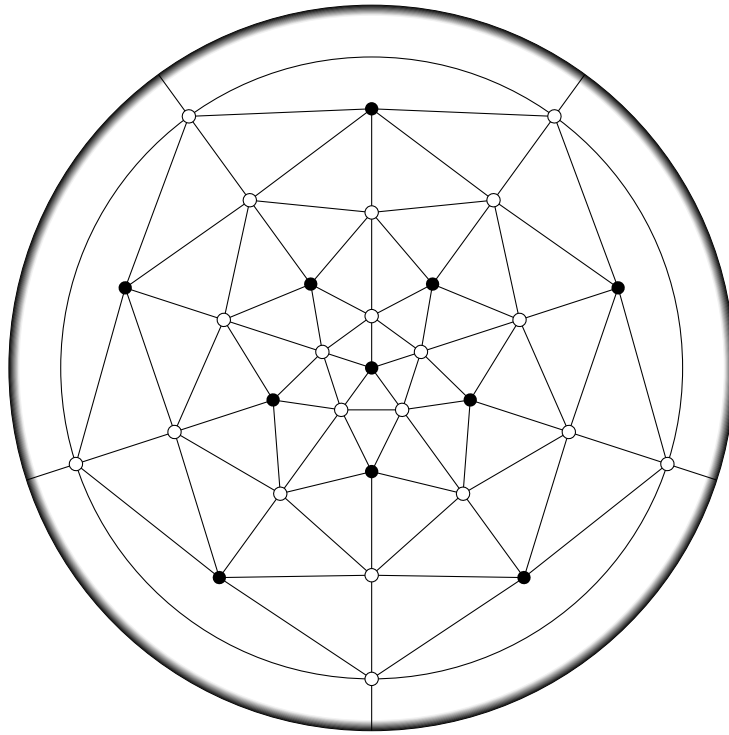
INTRODUCTION

A football pattern[1] is a graph embedded in the two-sphere in such a way that all faces are pentagons and hexagons, satisfying the conditions that the edges of each pentagon meet only edges of hexagons, and that the edges of each hexagon alternately meet edges of pentagons and of hexagons. If one requires that there are exactly three faces meeting at each vertex, then Euler's formula implies that the pattern consists of 12 pentagons and 20 hexagons. Moreover, in this case the combinatorics of the pattern is uniquely determined. This pattern, which we shall refer to as the standard football, has a particularly symmetric realisation with all polygons regular, which can be thought of as a truncated icosahedron.

If one drops the requirement that there are exactly three polygons meeting at each vertex, then one can exhibit infinitely many distinct football patterns by lifting to branched covers of the standard football branched only in vertices of the pattern. In the first part of this paper we shall prove that these are the only football patterns on the two-sphere. For the proof we consider the dual graph of a football pattern as a coloured ribbon graph. The dual of the standard football with 12 pentagons and 20 hexagons is shown in the picture on the first page of this paper. To make the picture symmetric, a black vertex is spread out at infinity and is depicted by the shaded ring around the rest of the graph.

Of course we could consider the dual graph as a map, but in order to make clear the distinction between football patterns and their duals, we will always use the language of maps, cf. [2], for the football patterns themselves, and the language of ribbon graphs, cf. [3], for their duals.

All the football graphs, that is ribbon graphs dual to football patterns, have the same universal covering, which is a certain tree T. We shall determine the automorphism group of T and prove that every cofinite subgroup of the automorphism group whose quotient gives rise to the dual of a spherical pattern is a subgroup of the group giving rise to the standard football. We also consider football patterns on surfaces of higher genera and show that the classification theorem does not hold for them; in other words, not all of them can be obtained by taking branched covers of the standard football. However, football patterns on surfaces of higher genera always admit branched covers of degree at most 60 which in turn are also branched covers of the standard spherical football.

One may wonder what special rôle pentagons and hexagons play in this discussion. We shall address this question in the second part of this paper, where we determine all the possibilities for triples (k, l, n) that can be realised by maps on the two-sphere whose faces are k-gons and l-gons satisfying the conditions that the edges of each k-gon meet only edges of l-gons, and that every n^{th} edge of each l-gon meets an edge of a k-gon, and its other edges meet l-gons. Not surprisingly, the determination of these triples is a generalisation of the topological argument determining the Platonic solids. In most cases the generalised football patterns have realisations dual to very symmetric triangulations of the sphere that have been known and studied since the 19th century. That purely topological or combinatorial considerations lead to a list that contains almost only the usual symmetric patterns and their degenerations is a kind of rigidity phenomenon associated with these spherical triangulations.

We shall see that the classification theorem for spherical football patterns proved in the first part of this paper for the triple $(5, 6, 2)$ actually holds for all generalised football patterns with $n = 2$: each generalized football with a pattern of type $(k, l, 2)$ is a branched cover of the corresponding minimal pattern. We shall also see that this result does not extend to $n > 2$.

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[1] We use English terminology. American readers might want to call our football patterns "soccer ball patterns".

1. RIBBON GRAPHS AND BRANCHED COVERS

1.1. **Football graphs.** A football pattern is a map in the sense of [2] on the two-sphere satisfying the usual conditions that at least three edges meet at every vertex, that all faces are pentagons and hexagons, that the edges of each pentagon meet only edges of hexagons, and that the edges of each hexagon alternately meet edges of pentagons and of hexagons. We make no regularity assumption, so that a football pattern is not a geometric, but a combinatorial-topological object.

A football pattern determines, and is determined by, its dual graph. This graph has a vertex for every polygon in the pattern, and the vertices are coloured, say black for the vertices corresponding to pentagons and white for the vertices corresponding to hexagons. Two vertices are connected by an edge if the corresponding polygons share an edge. The edges meeting at a vertex are cyclically ordered (with respect to this endpoint) by remembering that the sides of a polygon are cyclically ordered. Therefore the dual graph is a fatgraph or ribbon graph in the sense of [3], leading to the following definition:

Definition 1. A football graph is a ribbon graph with black and white vertices satisfying the following conditions:

(1) each black vertex has valence five, and all five edges connect the given vertex to white vertices, and
(2) each white vertex has valence six, and the six edges alternately[2] connect the given vertex to black and white vertices.

The picture at the beginning of this paper shows the dual ribbon graph of the standard football pattern on S^2 consisting of 12 pentagons and 20 hexagons. To make the picture symmetric, a black vertex is spread out at infinity and is depicted by the shaded ring around the rest of the graph.

As every finite ribbon graph corresponds to a unique closed oriented surface, we have a natural bijection between football graphs and football patterns on arbitrary closed oriented surfaces. A covering map between football graphs corresponds to a possibly branched covering map between surfaces, with any branching restricted to the centers of the faces of the decompositions given by the football graphs. As the football graphs are dual to actual football patterns, branching can only occur at the vertices of patterns.

Let b and w be the numbers of black and white vertices in a football graph Γ. Then the total number of vertices is $v = b + w$, and the number of edges is $e = \frac{1}{2}(5b + 6w)$. Of these edges, $e_1 = \frac{3}{2}w$ have white endpoints, and $e_2 = 5b = 3w$ have a black and a white endpoint. It follows that there is a natural number d such that $b = 6d$ and $w = 10d$.

Lemma 1. *The only football graph giving a triangulation of a closed surface is the dual graph Γ_0 of the standard spherical football with $d = 2$.*

Proof. Let Σ be the closed oriented surface defined by a football graph Γ. By Euler's formula, the number of faces in the cell decomposition of Σ determined by Γ is

$$f = \chi(\Sigma) + e - v = \chi(\Sigma) + 45d - 16d = \chi(\Sigma) + 29d \, .$$

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[2]The alternating condition is with respect to the cyclic order of the edges.

If Γ defines a triangulation of Σ, then we must have $2e = 3f$, which, rewritten in terms of d, means $90d = 3\chi(\Sigma) + 87d$, or $d = \chi(\Sigma)$. Thus $d = 2$, and Σ is S^2. The combinatorics of the corresponding football pattern is uniquely determined in this case, as can be seen from the proof of Theorem 2 below. $\qquad\square$

1.2. **The football tree.** There is precisely one connected and simply connected football graph, which we shall call the football tree T. It is the universal cover of any football graph. As T thought of as a ribbon graph is an orientable surface, it makes sense to speak of orientation-preserving automorphisms, and we shall denote the group of all such automorphisms by $\mathrm{Aut}(T)$. This can be determined explicitly:

Theorem 1. *The automorphism group* $\mathrm{Aut}(T)$ *of the football tree is isomorphic to the free product* $\mathbb{Z}_2 \star \mathbb{Z}_3 \star \mathbb{Z}_5$.

Proof. This is a straightforward application of the Bass–Serre theory of groups acting on trees, cf. [4]. This theory is usually formulated for groups acting on trees without inverting edges. In our situation, there are automorphisms inverting edges that connect a pair of white vertices. Therefore, we subdivide each of these edges by introducing red vertices in the middle of each edge connecting two white edges of T. We obtain a new tree T', which has three kinds of edges: the white and black ones of valence 6 and 5 respectively, and red ones of valence 2. The red and black vertices are only connected to white ones, and the edges meeting at a white vertex lead alternately to red and black vertices. Now $\mathrm{Aut}(T)$ acts on T' without inverting any edges. The action is simply transitive on the two kinds of edges, black-white and red-white. The action is also transitive on the vertices of a given colour, with isotropy groups of orders 2, 3 and 5 for the red, white and black vertices. The quotient graph $T'/\mathrm{Aut}(T)$ is a tree with three vertices, one for each colour, and with one edge connecting the white vertex to each of the other vertices. We think of this as a graph of groups by labeling the vertices with the isotropy groups. As the edges have trivial isotropy, the fundamental group of this graph of groups is the free product of the labels of the vertices. By the structure theorem of Section I.5.4 in [4], this fundamental group is isomorphic to $\mathrm{Aut}(T)$. $\qquad\square$

1.3. **The classification of spherical footballs.** We now want to prove that every football pattern on the sphere is obtained as a branched cover of the standard football branched only in the vertices. Equivalently, we prove that the spherical dual ribbon graphs are all obtained as covering spaces of the dual graph of the standard football. The first step is the following:

Lemma 2. *Let* Γ *be a football graph with universal covering* $\pi\colon T \longrightarrow \Gamma$. *Suppose that* γ *is an oriented path in* T *consisting of a sequence of edges without backtracking. If* π *maps* γ *to a closed path that corresponds to a boundary component of* Γ *thought of as a surface with boundary, then* γ *consists of* $3n$ *edges for some natural number* n, *and the image of* γ *in the standard football graph* Γ_0 *is a loop that is the* n^{th} *power of the loop formed by a triangle in the triangulation defined by* Γ_0, *cf. Lemma 1.*

Proof. We think of T as a surface with boundary. Choose a boundary component C covering the boundary component $\pi(C)$ of Γ to which γ is mapped. As a boundary component of T, C runs right along a sequence $\{e_i\}_{i\in\mathbb{Z}}$ of oriented edges in T such that, of course, the origin $o(e_i)$ of each edge coincides with the endpoint of the previous edge, and, in addition, with respect to the cyclic order of edges emanating from $o(e_i) = o(\bar{e}_{i-1})$, e_i is the successor of $\bar{e}_{i-1}$. (The bar denotes edge inversion.) It follows from Definition 1 that the sequence of vertices $o(e_i)$ is of the form *black, white, white, black, white, white*, etc.

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The setwise stabilizer of C in $\mathrm{Aut}(T)$ is the infinite cyclic group generated by the translation τ, which maps e_i to e_{i+3}. Now $\pi(C) = C/G$, for some subgroup $G \leqslant \mathrm{Stab}(C)$, i.e. $G = \langle \tau^n \rangle$ for some $n \in \mathbb{N}$. Thus $\pi(C)$ runs along a $3n$-gon, which must be $\pi(\gamma)$.

By assumption, the path γ is a piece $e_{j+1}, \ldots, e_{j+k}$ of the sequence $\{e_i\}$ and projects to $\pi(\gamma)$. Therefore $k = 3n$. The images of e_i in the standard football still satisfy the condition that consecutive edges be related by the cyclic ordering defining the ribbon graph structure. But this means that the image of $e_{j+1}e_{j+2}e_{j+3}$ is a triangle. $\qquad\square$

Here is the classification theorem for spherical football patterns:

Theorem 2. *Every football graph dual to a football pattern on S^2 is a finite covering space of the standard football graph Γ_0. Equivalently every football pattern on S^2 is obtained from the standard one by passing to a branched cover branched only in vertices of the pattern.*

Proof. Given a spherical football graph Γ, fix a universal covering $\pi\colon T \to \Gamma$ of Γ and a universal covering $\pi_0\colon T \to \Gamma_0$ of the standard football graph Γ_0. We are going to show that the group of deck transformations $\mathrm{Aut}_\Gamma T$ is a subgroup of $\mathrm{Aut}_{\Gamma_0} T$. This implies that π_0 factors through π.

Choose a point p on an edge of T that is not an endpoint or a midpoint, so that it has trivial stabilizer in $\mathrm{Aut}(T)$. Covering space theory identifies the group of deck transformations $\mathrm{Aut}_\Gamma T$ with the fundamental group $\pi_1(\Gamma; \pi(p))$. Since Γ is spherical, this fundamental group is generated by paths of the form $\beta\gamma\beta^{-1}$, where γ is a loop along a boundary component, β runs from the base point $\pi(p)$ to the origin of γ and β^{-1} is the way back. Lifting β and γ to T we obtain a path $\tilde{\beta}\tilde{\gamma}\tau^n(\tilde{\beta}^{-1})$ leading from p to $\tau^n(p)$, with τ from the proof of the previous lemma. Hence τ^n is the deck transformation corresponding to the given generator of $\pi_1(\Gamma; \pi(p))$. This proves the result, because τ is a deck transformation over Γ_0. $\qquad\square$

This proof also shows:

Corollary 1. *The subgroup $\pi_1(\Gamma_0) \subset \mathrm{Aut}(T)$ is normal. The quotient $\mathrm{Aut}(T)/\pi_1(\Gamma_0)$ is the icosahedral group of order 60.*

1.4. **Footballs of positive genera.** Every ribbon graph corresponds to a unique closed oriented surface, and of course every such surface does indeed arise from a football graph, for example because it is a branched covering of the two-sphere, which we can arrange to be branched only in the vertices of a suitable football pattern. We now want to show that there are other football patterns on surfaces of positive genera, that are not lifted from the two-sphere. The proof of Theorem 2 does not extend, because for a ribbon graph corresponding to a surface of positive genus there are generators in the fundamental group that arise from handles, rather than from the punctures.

Recall that the parameter d for a finite football graph specifies the number of black and white vertices by the formulae $b = 6d$ and $w = 10d$. Passing to a D-fold covering multiplies d by D. As the standard football graph Γ_0 has $d = 2$, all its non-trivial coverings have $d \geqslant 4$. Therefore, to exhibit football graphs that are not coverings of Γ_0, it suffices to find examples of positive genus with $d < 4$. Performing certain cut-and-paste operations on Γ_0, we can actually produce examples with $d = 2$ and rather large genera.

The simplest example is the following.

Example 1. Pick two disjoint edges in the standard football pattern, that are of the same type, so that they both separate hexagons from each other, or they both separate a pentagon from a hexagon. Open up the two-sphere along these edges to obtain a cylinder whose two boundary circles each

have two vertices and two edges. As the two edges along which we opened the sphere were of the same type, the two boundary circles of the cylinder can be identified in such a way that the resulting torus carries an induced football pattern with $d = 2$.

In this example there are 58 vertices instead of the 60 in the standard spherical football. All but two of them are 3-valent, and the exceptional two are 6-valent.

In the language of ribbon graphs, the surgery performed in the above example amounts to cutting two ribbons and regluing the resulting ends in a different pairing. This can also be done with ribbons corresponding to edges that share a vertex, in which case instead of cutting and pasting, the surgery can be described through the reordering of edges:

Example 2. Let $e_1, \dots, e_5$ be the edges emanating from a black vertex in the standard football graph Γ_0. Define a new football graph Γ by reordering the edges as e_1, e_3, e_2, e_4, e_5. This procedure glues the three triangles whose edges include e_2 or e_3 into a single 9-gon boundary component of Γ. This new graph still has $d = 2$, but the underlying surface is a torus. In the dual football pattern there are 58 vertices, of which 57 are 3-valent and one is 9-valent.

Example 3. Let $e_1, \dots, e_6$ be the edges emanating from a white vertex in the standard football graph Γ_0, enumerated in their cyclic order and labelled such that e_1, e_3 and e_5 have black ends. Define a new ribbon graph by reordering the edges cyclically as $e_1, e_4, e_3, e_6, e_5, e_2$. This means that the edges leading to white vertices are cut and reattached after a cyclic permutation given geometrically by a rotation by angle $\frac{2\pi}{3}$ around the vertex in the realisation of Γ_0 with icosahedral symmetry.

We apply this procedure to every white vertex of Γ_0. The resulting football graph Γ is symmetric in the sense that it admits rotations of order 2, 3 and 5 around an edge, a white vertex and a black vertex, respectively. Hence the full football group $\mathrm{Aut}(T)$ acts on Γ. In particular all faces are conjugate. One can verify by inspection that the faces are 15-gons. Hence the Euler characteristic of the underlying surface is -46, and its genus is 24. This is a football graph with $d = 2$ and is therefore not a covering of Γ_0.

Although football graphs of positive genera are not in general coverings of Γ_0, we have the following:

Theorem 3. *Every football pattern on a closed oriented surface admits a branched cover of degree $D \leqslant 60$ that is also a branched cover of the standard minimal pattern on S^2. The bound for D is sharp.*

Proof. Let Γ be a finite football graph. As $\pi_1(\Gamma_0) \subset \mathrm{Aut}(T)$ is a subgroup of index 60, the intersection $\pi_1(\Gamma_0) \cap \pi_1(\Gamma)$ has index at most 60 in $\pi_1(\Gamma)$. The intersection corresponds to a covering of Γ of degree $D \leqslant 60$ that is also a covering of Γ_0.

To prove that coverings of degree strictly less than 60 do not always suffice, recall that $\mathrm{Aut}(T)$ acts on the genus 24 football graph Γ from Example 3. This is equivalent to the fundamental group $\pi_1(\Gamma)$ being a normal subgroup of $\mathrm{Aut}(T)$. The covering of Γ corresponding to the subgroup $N = \pi_1(\Gamma_0) \cap \pi_1(\Gamma)$ of $\mathrm{Aut}(T)$ is a Galois covering with Galois group $\pi_1(\Gamma)/N$. Now the injection $\pi_1(\Gamma) \to \mathrm{Aut}(T)$ induces an embedding of $\pi_1(\Gamma)/N$ as a normal subgroup of the orientation-preserving icosahedral group $\mathrm{Aut}(T)/\pi_1(\Gamma_0)$, isomorphic to the alternating group A_5. Since this is a simple group we must have $\pi_1(\Gamma)/N \cong A_5$ or $\{1\}$. The second case is excluded because Γ is not a covering of Γ_0. $\qquad\square$

1.5. **Non-orientable footballs.** Although we have modelled football patterns as ribbon graphs, we can also consider them on non-orientable surfaces, because the condition that every other edge emanating from a white vertex should connect to a black vertex is preserved by inversion of the cyclic order.

Here are the simplest examples for the projective plane.

Example 4. A football pattern on the real projective plane is readily constructed from the standard football. In the dual ribbon graph Γ_0 cut a single ribbon and reglue it with a half-twist so that the surface becomes non-orientable. This gives a football pattern with $d = 2$ that, instead of the 60 vertices of valence 3 in the standard football, has 58 vertices of valence 3 and a unique vertex of valence 6. Therefore the Euler number of the surface is 1.

If we lift the pattern in this example to the universal covering of the projective plane, we obtain a football pattern with $d = 4$ on the two-sphere, which, by Theorem 2, is a 2-fold branched cover of the standard pattern. Of course in this case the branched covering structure can be seen directly, by focussing on the two vertices of valence 6.

Example 5. The usual symmetric realisation of the standard football pattern on the sphere as a truncated icosahedron is symmetric under the antipodal involution. Thus it descends to a pattern on $\mathbb{R}P^2$ with $d = 1$.

2. GENERALISED FOOTBALL PATTERNS

In this section we consider generalisations of the traditional football patterns.

Definition 2. A generalised football pattern is a map on the two-sphere whose faces are k-gons and l-gons satisfying the conditions that the edges of each k-gon meet only edges of l-gons, and that every n^{th} edge of each l-gon meets an edge of a k-gon, and its other edges meet l-gons.

To avoid degenerate cases we always assume $k \geqslant 3$, $l \geqslant 3$ and $l = n \cdot m$ with positive integers m and n. As usual, at least three edges meet at every vertex.

If a given triple (k, l, n) can be realised by a generalised football pattern, then it can be realised in infinitely many ways, for example by taking branched covers branched only in the vertices of a given pattern. We will determine all possible triples, and we will find a minimal realisation for each of them. We will also see that in some cases there are realisations that are not branched covers of the minimal one.

Before proceeding to the classification, we list some examples for future reference.

2.1. **Some examples.** The standard football realising the triple $(5, 6, 2)$ can be thought of as a truncated icosahedron. More generally, we have:

Example 6. The truncated Platonic solids realise the triples $(3, 6, 2)$, $(3, 8, 2)$, $(3, 10, 2)$, $(4, 6, 2)$ and $(5, 6, 2)$ as generalised football patterns.

There are also infinite series of examples obtained by truncating the degenerate Platonic solids:

Example 7. Start with a subdivision of the sphere along $k \geqslant 3$ halves of great circles running from the north to the south poles. We shall call this an American football. If we now truncate at one of the poles, we obtain a realisation of the triple $(k, 3, 3)$. (For $k = 3$ this is a tetrahedron.) If we truncate at both poles, we obtain a tin can pattern realising $(k, 4, 2)$. (For $k = 4$ it is a cube.) This is also known as a k-prism. If we add k edges along the equator to this last example, we obtain a double tin can realising $(k, 4, 4)$, for any $k \geqslant 3$.
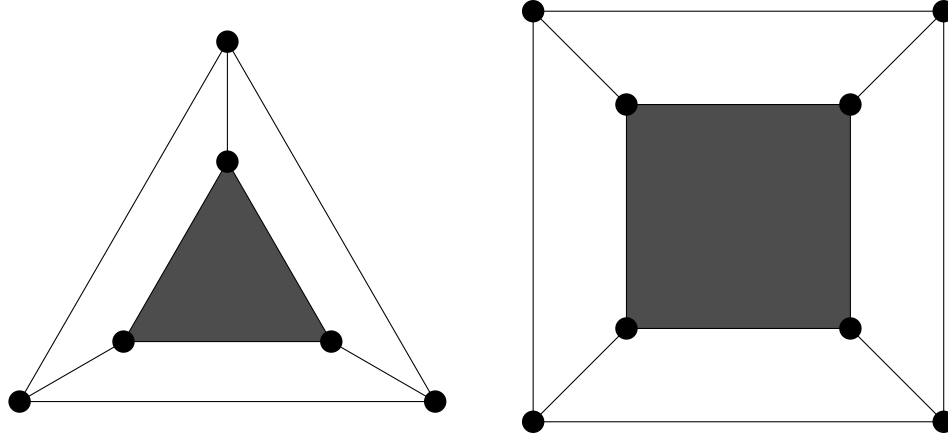
FIGURE 1. Variation on Platonic solids

Here is a variation on the above tin can pattern:

Example 8. Take a k-gon, with $k \geqslant 3$ arbitrary, and surround it by pentagons in such a way that the two pentagons meeting a pair of adjacent sides of the k-gon share a side. We can fit together two such rings made up of a k-gon and k pentagons each along a zigzag curve to obtain a realisation of the triple $(k, 5, 5)$ by a generalised football pattern. (For $k = 5$ we obtain a dodecahedron.)

The next two examples need to be visualised using the accompanying figures.

Example 9. Take a Platonic solid, and subdivide each face as follows. If the face is a k-gon, put a smaller k-gon in its interior, and radially connect each corner of this smaller k-gon to the corresponding corner of the original face. In this way each face of the original polyhedron is divided into a k-gon and k quadrilaterals. The cases $k = 3$ and 4 are shown in Figure 1. Next we erase the edges of the original polyhedron, so that the two quadrilaterals of the subdivison meeting along an edge are joined to form a hexagon. In this way we obtain a realisation of the triple $(k, 6, 3)$. We shall refer to this construction as a variation on the original Platonic solid.

Example 10. Again we start with a Platonic solid whose faces are k-gons. We subdivide each face into a smaller k-gon and k hexagons as shown in Figure 2. This gives a realisation of the triple $(k, 6, 6)$.

2.2. **The classification of generalised football patterns.** Now we want to prove that the previous examples exhaust all possible generalised football patterns with $n \geqslant 2$. We shall also treat the case $n = 1$. The results of this classification are summarised in the table in Figure 3.

If S^2 is endowed with a generalised football pattern of type (k, l, n), we shall think of the k-gons as being black and the l-gons as being white. Their numbers are denoted by b and w respectively. The pattern then has $f = b + w$ many faces, it has $e = \frac{1}{2}(bk + wl)$ many edges, and the number v of vertices is bounded by $v \leqslant \frac{1}{3}(bk + wl)$ as there have to be at least 3 faces meeting at every vertex.

Counting the number of edges at which a k-gon meets an l-gon in two different ways, we find

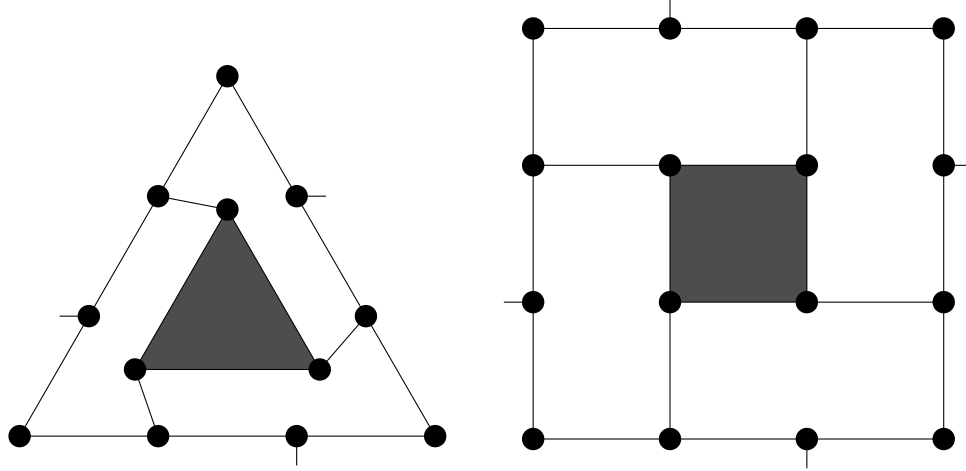(1) $$k \cdot b = \frac{1}{n} \cdot l \cdot w = m \cdot w \,.$$

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FIGURE 2. Subdivision of Platonic solids

Computing the Euler characteristic and using (1) leads to:

$$2 = f - e + v \leqslant b + w - \frac{1}{6}(k \cdot b + l \cdot w) = b + w - \frac{1}{6}k \cdot b \cdot (n+1) \, .$$

Dividing by $2k \cdot b$ we obtain the key inequality

(2)
$$\frac{1}{k \cdot b} + \frac{n+1}{12} \leqslant \frac{1}{2k} + \frac{1}{2m} \, .$$

2.2.1. *The case $n \geqslant 2$.* If $n \geqslant 2$, then the left hand side of (2) is strictly larger than $\frac{1}{4}$. If both k and m are at least 4, then the right hand side is at most $\frac{1}{4}$. Thus $n \geqslant 2$ implies $k = 3$ or $m \in \{1,2,3\}$. We now discuss these cases separately.

For every possible triple (k, l, n) the inequality (2) gives a lower bound for b. With (1) we then have a lower bound for w. Checking these values against the examples of the previous section, one sees for most of the examples that they actually give minimal realisations. There are only a few cases when this simple check does not suffice, because the minimal realisations have vertices of valence strictly larger than 3.

Lemma 3. *Suppose the triple (k, l, n) with $k = 3$ and $n \geqslant 2$ is realised by a generalised football pattern. Then (k, l, n) is one of the triples $(3, 4, 2)$, $(3, 6, 2)$, $(3, 8, 2)$, $(3, 10, 2)$, $(3, 3, 3)$, $(3, 6, 3)$, $(3, 4, 4)$, $(3, 5, 5)$, $(3, 6, 6)$. Minimal realisations are given by the Examples 6, 7, 8, 9 and 10.*

Proof. Putting $k = 3$ in (2) we obtain

(3)
$$\frac{1}{3b} + \frac{n}{12} \leqslant \frac{1}{12} + \frac{1}{2m} \, .$$

Together with $n \geqslant 2$, this implies $m \leqslant 5$.

If $m = 5$, then we obtain $n = 2$ from (3). Thus $l = 10$. The truncated dodecahedron from Example 6 is a minimal realisation.

If $m = 4$, then again $n = 2$ from (3). Thus $l = 8$. The truncated cube from Example 6 is a minimal realisation.

If $m = 3$, then again $n = 2$ from (3). Thus $l = 6$. The truncated tetrahedron from Example 6 is a minimal realisation.

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If $m = 2$, then (3) gives $n \leqslant 3$. For $n = 3$, equivalently $l = 6$, a minimal realisation is the variation on the tetrahedron from Example 9. For $n = 2$, equivalently $l = 4$, we get the case $k = 3$ of the truncated American football in Example 7.

If $m = 1$, then (3) only gives $n \leqslant 6$. For $n = 6$, therefore $l = 6$, the subdivision of the tetrahedron from Example 10 gives a minimal realisation. For $n = l = 5$, the case $k = 3$ in Example 8 is a minimal realisation. For $n = l = 4$, the case $k = 3$ of the double tin can in Example 7 is a realisation. As it has vertices of valence 4, it is not immediately obvious that it is a minimal realisation. In this case (3) gives $b \geqslant 1$, but one can easily check that $b = 1$ is not possible. Thus b is at least 2, and the double tin can is indeed a minimal realisation. For $n = l = 3$, the case $k = 3$ of the partially truncated American football in Example 7 is a minimal realisation. Note that $m = 1$ implies $n \geqslant 3$, so that we have now exhausted all cases with $k = 3$ and $n \geqslant 2$. $\qquad\square$

Lemma 4. *Suppose the triple (k, l, n) with $l/n = m = 3$ and $n \geqslant 2$ is realised by a generalised football pattern. Then (k, l, n) is one of the triples $(3, 6, 2)$, $(4, 6, 2)$, $(5, 6, 2)$. Minimal realisations are given by the truncated Platonic solids in Example 6.*

Lemma 5. *Suppose the triple (k, l, n) with $l/n = m = 2$ and $n \geqslant 2$ is realised by a generalised football pattern. Then (k, l, n) is one of the triples $(3, 6, 3)$, $(4, 6, 3)$, $(5, 6, 3)$, or $(k, 4, 2)$ with $k \geqslant 3$. Minimal realisations are given by the variations on the Platonic solids in Example 9, respectively by the truncated American football in Example 7.*

We omit the proofs of Lemmas 4 and 5, because they are completely analogous to, and even simpler than, the proof of Lemma 3.

Lemma 6. *Suppose the triple (k, l, n) with $l/n = m = 1$ and $n \geqslant 2$ is realised by a generalised football pattern. Then (k, l, n) is one of the triples $(3, 6, 6)$, $(4, 6, 6)$, $(5, 6, 6)$, or $(k, 3, 3)$, $(k, 4, 4)$, $(k, 5, 5)$ with $k \geqslant 3$. Minimal realisations are given by the subdivisions on the Platonic solids in Example 10, respectively by the infinite sequences in Examples 7 and 8.*

Proof. For $m = 1$, equivalently $l = n$, we obtain $n \leqslant 6$ for all $k \geqslant 3$ from (2).

If $l = n = 6$, then we also have $k \leqslant 5$ from (2). Thus k is 3, 4 or 5, and minimal realisations are given in Example 10.

If $l = n = 5$, then all $k \geqslant 3$ are possible, and minimal realisations are given in Example 8.

If $l = n = 4$, then all $k \geqslant 3$ are possible, and realisations are given by the double tin cans in Example 7. To see that these realisations are minimal, it suffices to exclude the case $b = 1$, which is easily done by contradiction.

If $l = n = 3$, then again all $k \geqslant 3$ are possible, and minimal realisations are given by the partially truncated American football in Example 7. $\qquad\square$

This completes the classification of generalised football patters with $n \geqslant 2$ on the two-sphere.

2.2.2. *The case $n = 1$.* A generalised football pattern with $n = 1$ consists of b black k-gons and w white l-gons so that the two polygons meeting along an edge always have different colours. Note that here the situation is completely symmetric in k and l.

Lemma 7. *Suppose the triple $(k, l, 1)$ is realised by a generalised football pattern on S^2. Then, up to changing the rôles of k and l, (k, l) is one of the pairs $(3, 3)$, $(3, 4)$ or $(3, 5)$. Minimal realisations are obtained by painting the faces of an octahedron, a cuboctahedron, respectively an icosidodecahedron, in a suitable manner.*

Proof. Counting the edges in two different ways leads to $b \cdot k = w \cdot l$. As every edge separates a black from a white polygon, there must be an even number of faces meeting at every vertex. Therefore the valence of every vertex is $\geqslant 4$, giving rise to

$$v \leqslant \frac{1}{4}(b \cdot k + w \cdot l) \,,$$

which is of course stronger than what we had before, when the valence of a vertex was only $\geqslant 3$. Computing the Euler characteristic as before, we obtain instead of (2) the stronger

$$(4) \qquad\qquad \frac{1}{k \cdot b} + \frac{1}{4} \leqslant \frac{1}{2k} + \frac{1}{2l} \,.$$

If both k and l are $\geqslant 4$, then the right hand side is $\leqslant \frac{1}{4}$, which is impossible. Thus k or l is $= 3$. By the symmetry between k and l we may assume that $k = 3$. Substituting this into (4), we find

$$(5) \qquad\qquad \frac{1}{3b} + \frac{1}{12} \leqslant \frac{1}{2l} \,,$$

which implies $l \leqslant 5$.

If $l = 3$, then $b = w \geqslant 4$. A realisation is obtained by painting the faces of an octahedron in black and white, so that each edge separates a black face from a white one.

If $l = 4$, then (5) implies $b \geqslant 8$, so that $w \geqslant 6$. The cuboctahedron, cf. [1], is a realisation.

If $l = 5$, then (5) implies $b \geqslant 20$, so that $w \geqslant 12$. The icosidodecahedron, cf. [1], is a realisation.

All these realisations are minimal, because they have precisely four faces meeting at every vertex, so that (5) becomes an equality. $\qquad\square$

We summarise the above classification of the generalised football patterns as follows:

Theorem 4. *Suppose that S^2 admits a map whose faces are k-gons and l-gons with $k, l \geqslant 3$, such that the edges of each k-gon meet only edges of l-gons, and so that every n^{th} edge of each l-gon meets an edge of a k-gon, and its other edges meet l-gons. Then $l \leqslant 10$ and $n \leqslant 6$. There are 16 different sporadic triples (k, l, n) with $k \leqslant 5$ that occur, together with 4 infinite sequences with variable k and fixed l and n. All the possibilities are listed in the table in Figure 3, which also gives minimal realisations for all cases.*

The different minimal realisations were described in our earlier examples, and in the course of the proof. There are several alternative descriptions of items 12.–14. The variation on the tetrahedron is nothing but a partially truncated cube, truncated at 4 of its 8 vertices, chosen so that each face is truncated at two diagonally opposite corners. Similarly, the variations on the cube and the dodecahedron are partial truncations of the rhombic dodecahedron and of the rhombic triacontahedron respectively. The subdivisions in items 18.–20. can also be thought of as partial truncations of the dodecahedron, the pentagonal icositetrahedron, respectively the pentagonal hexecontahedron.

In order to make the symmetries more obvious, the table does not list triples (k, l, n), but rather (k, m, n) with $m = l/n$. Note that this makes no difference when $n = 1$, in which case there is a complete symmetry between k and $l = m$. Therefore, the cases 2. and 3., respectively 4. and 5., are dual to each other, obtained by switching the roles of k and l. Case 1. is self-dual. Similarly, cases 7. and 8., respectively 9. and 10., are dual to each other with the duality induced by the duality of Platonic solids, and case 6. is self-dual.

	k	m	n	minimal realisation	b	w
1.	3	3	1	octahedron	4	4
2.	3	4	1	cuboctahedron	8	6
3.	4	3	1	cuboctahedron	6	8
4.	3	5	1	icosidodecahedron	20	12
5.	5	3	1	icosidodecahedron	12	20
6.	3	3	2	truncated tetrahedron	4	4
7.	3	4	2	truncated cube	8	6
8.	4	3	2	truncated octahedron	6	8
9.	3	5	2	truncated dodecahedron	20	12
10.	5	3	2	truncated icosahedron = **football**	12	20
11.	$\geqslant 3$	2	2	truncated American football	2	k
12.	3	2	3	variation on the tetrahedron	4	6
13.	4	2	3	variation on the cube	6	12
14.	5	2	3	variation on the dodecahedron	12	30
15.	$\geqslant 3$	1	3	partially truncated American football	1	k
16.	$\geqslant 3$	1	4	double tin can	2	2k
17.	$\geqslant 3$	1	5	zigzag tin can	2	2k
18.	3	1	6	subdivision of the tetrahedron	4	12
19.	4	1	6	subdivision of the cube	6	24
20.	5	1	6	subdivision of the dodecahedron	12	60

FIGURE 3. The classification of generalised football patterns on S^2

2.3. **Branched covers for generalised football patterns.** Now that we have an overview of all the generalised football patterns, one may ask whether Theorem 2 can be extended, to prove that every generalised spherical football is a branched cover of the corresponding minimal example, branched only in the vertices. It is not hard to see that for $n = 2$ this is indeed the case:

Theorem 5. *If $(k, l, 2)$ is realised by a generalised football pattern on S^2, then every spherical realisation is a branched cover, branched only in the vertices, of the minimal realisation. In particular the minimal realisation is unique.*

Proof. The minimal realisations for $n = 2$ have all vertices of valence 3. Equivalently, the dual graph triangulates the sphere. Moreover, every realisation has the property that the valence of every vertex is a multiple of 3, with every third face a black k-gon. Therefore the proofs of Lemma 2 and of Theorem 2 go through. □

For larger values of n one loses control of the structure of the vertices, and the proof breaks down.

Example 11. Consider the case $k = l = n = 3$. The minimal realisation is a partially truncated American football with $k = 3$, which we can also think of as a painted tetrahedron, in which one face is black and the others are white. Another, non-minimal, realisation is obtained by painting the faces of an octahedron so that two opposite faces are black, and the remaining six faces are white. This is not a branched cover of the painted tetrahedron.

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Note that in this example the minimal realisation has two kinds of vertices: one at which all faces meeting there are white, and three at which one black and two white faces meet. For the non-minimal realisation described above one has one black and three white faces meeting at every vertex. It is a general feature of the generalised football patterns with $n \geqslant 3$ that the combinatorial definition of the pattern does not imply any control over the local structure at a vertex. For $n = 2$ we do have such control, leading to Theorem 5.

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